Research Proposal
Stochastic methods in quantum systems

November 17, 2022

The present project aims at applying and further develing techniques of stochastic analysis in relation with the study of quantum systems, both in the finite and in the infinite dimensional case. The project constists of two parts: the first part concentrates on problems of quantum fields and associated stochastic partial differential equations (SPDEs) and variational problems. Particular attention is given to extend methods already developed for scalar quantum fields to the case of Fermionic quantum fields, using in particular a Grassmanian algebraic setting. Analytic methods of Dirichlet type operators and Hamiltonian methods will play an important role, extended to the non commutative setting. Also functional analytic methods going back to fundamental work of Nelson, developed in connection with his view of the role of probabilistic methods for quantum theory and quantum fields. The second part of the project aims at developing related probabilistic ideas and methods for the study of nonlinear PDEs, like the Gross-Pitaevskii equation, arising in the study of a scaling limit of certain coupled quantum mechanical N particle systems, in the large N limit. The probabilistic methods are inspired by Nelson's stochastic mechanics, as in the first part of the project. New variational principles will also play an important role in both parts of the project ([62]).

1 Interacting Bosons and Fermions: stochastic PDEs and variational problems

Probabilistic methods in quantum field theory (QFT) have proved to be particularly fruitful (cf. e.g. [18, 23, 37]). These methods have been almost exclusively restricted to Bosonic field theories. Some ideas of the Bosonic probabilistic methods carry over, to an extent, to the Fermionic case using the beautiful algebraic technique of Berezin integration [6]. This extension in conjunction with renormalization group techniques was successfully used to build some quantum field models involving interacting Fermions (see, e.g., [29] and references therein, [30],[31]).

More recently, in [1] (see also [10]), a framework, closer to the probabilistic approach used for Bosonic fields, was presented to describe Fermionic systems. This framework offers a new language to reformulate the ideas of Berezin in a more probabilistic and C^*-algebraic flavor. We think this framework to be very natural and to adapt the methods and ideas of quantization of Bosonic fields to the case of quantum systems involving Fermions (in particular the ideas of *stochastic quantization* introduced by Parisi-Wu [35] and recently applied to many cases thanks to the methods of singular stochastic partial differential equations (SPDEs) (see, e.g., [**?**], [9, 19, 22, 24])).

We aim to extend and apply the cited framework to the construction of interacting Fermionic quantum field theories in at least three directions.
The first is the generalization to the Fermionic case of the variational methods for (Bosonic) quantum field theory introduced by E. Nelson in finite dimension (see, e.g., [32, 33] and the references cited in the first part) and applied to infinite dimensional quantum fields in, e.g., [3, 4]. Starting from the results in [10] where a formulation using backward-forward anti-commutative SDEs is used, we would like to give a complete variational formulation of Euclidean Fermionic fields and apply it to critical models.
The second direction consists in applying the mentioned probabilistic method in the construction of the Hamiltonian of Fermionic fields on the space generated by the Fermionic fields at time zero. In this way the unitary time evolution of the regularized Fermionic models on the Fermionic Fock space, over Minkowski space-time, can be converted to a stochastic evolution (in the sense of [1]) in the Euclidean space-time. After removal of the regularization

this would constitute an improvement upon the previous results in [34, 13]. This treatment of operators in the Minkowski space time is also closely related to recent works [21, 20] in the context of SPDEs. As an application, we intend to reformulate within this language the fundamental work [17] regarding the Hamiltonian approach to the construction of the Yukawa$_2$ model. More generally, our goal is to employ the mentioned techniques to study systems of interacting Bosons and Fermions (see, e.g., [7, 26] and references in [1]).
The third application is in the direction of non-relativistic systems. Indeed, while this framework can be formulated ab initio in an abstract way, at its core, it is in close relation with the classical framework of quantum fields developed in theoretical and mathematical physics. Hence we can easily make use of powerful ideas coming from these disciplines, ideas which have deep physical and mathematical meaning. In particular we have in mind some methods and results within the context of the study of many body systems and multiscale analysis. In the case of many body systems, we would like to mention in particular the works [12, 28, 27]. Regarding multiscale analysis and in particular the renormalization group, we would like to mention the already cited [26] and also [11, 15, 16, 24]. We also plan to take advantage of results regarding current algebras and the implementability of Bogoliubov transformations (cf. e.g. [8]), and results regarding Schwinger terms and anomalies (regarding Schwinger terms cf. [14, 25], regarding anomalies cf. [5, 36]). These properties have a structural nature which should appear naturally within our framework, and so get a new interpretation.

2 Stochastic description of Bose-Einstein Condensation and the Gross-Pitaevskii model: new developments.

Bose-Einstein Condensation (BEC) is a purely quantum phenomenon, predicted by Einstein in 1925 at theoretical and mathematical level, on the basis of the article by the Indian physicist S.N. Bose [38], in the form of the occurrence of a transition phase in a gas of non-interacting atoms. The experimental realization, provided only in 1995 by E.Cornell and C. Wieman [39], who won, jointly with Wolfgang Ketterle, the Nobel Prize in Physics 2001 ''for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates", aroused a great interest in the mathematical description of this quantum phenomenon and in particular for its rigorous justification starting from first principles, that is from the system of N interacting bosons and by performing a suitable infinite particles limit. Standard Quantum Mechanics uses analytical mathematical techniques to describe this phenomenon and in particular the motion of the single particle in the BE condensate is described through a wave function which minimizes a non-linear functional called Gross-Pitaevskii (GP) functional. A recent physical-mathematical formalization of the phenomenon was provided between 2000 and 2002 by Lieb, Seiringer and Yngvason [40]. It considers charged quantum particles confined in a bounded region by a magnetic field acting on it. This non-linear one-particle model was (analytically) justified by Lieb and collaborators (2002) starting by the N-body Hamiltonian, calculating its ground state energy and making a limit of infinitely many particles, with an appropriate rescaling of the interaction potential, known as the GP scaling limit (see [40]). There is also a large community working on other scaling limits such as the mean-field scaling limit and intermediate scaling limit, giving rise asymptotically to the simpler Hartree equation and the non-linear Schroedinger equation, respectively (see, e.g., [41]). A stochastic approach has been proposed in 2011 ([42]), based on Nelson's Stochastic Mechanics ([43], see [44] for a more recent review), in which a one particle stochastic process is rigorously associated to the N-body Hamiltonian which, within the GP scaling limit, remains outside a time-dependent interaction region with probability one. This process is not Markovian but, when appropriately stopped, it converges in total variation in the GP scaling limit. The proof is based on Girsanov theorem and a relative entropy approach. In [45] a related result of localization of relative entropy in the same GP limit is described. In 2012 Ugolini establishes the Kac's chaos (or transition to chaos) for the probability law of the system of N interacting diffusions associated with the ground state of the N-body Hamiltonian in the GP scaling limit ([46]). It means that the interacting diffusions became asymptotically independent in the sense that for all n the n-particle probability density factorized in the product of n copies of the same probability density, which is the square of the minimizer of the GP non-linear functional. Furthermore, in 2015 Albeverio, Ugolini correctly identify the stochastic process associated with the minimizer of the (non-linear) functional of GP ([47]). A diffusion generator with a killing rate governed by the wave function of the condensate was determined. The killing rate, which depends on the probability density of the process itself, makes the process highly non-linear and represents the probabilistic way of describing the self-interaction undergone by the diffusion process of the single particle by the generic other single particle in the condensate. By introducing a suitable one-particle relative entropy in 2014, De Vecchi and Ugolini ([48]) prove an existence theorem for the probability measure associated with the GP minimizer. In 2017 Albeverio, De Vecchi and Ugolini ([49]) establish that the property of entropy chaos holds (a recent concept introduced in 2014 by Carlen et al. in [50]) for the aforementioned system of N interacting diffusions and they prove the weak convergence of the probability measure of the one-particle diffusion on the path space. The last is non-trivial problem due to the fact that although the interaction between the particles is asymptotically concentrated in a random region with a zero Lebesgue measure, it nevertheless does not disappear. The result was obtained using a relevant property of the sequence of stopping times. This weak convergence result provides the first probabilistic justification of the mathematical (analytical) model of Gross-Pitaevskii equation for the Bose-Einstein Condensate. In the case of the mean-field approach, a stronger type convergence has been obtained which allows us to prove that instead of the usual propagation of chaos, the strong Kac's chaos is valid ([51]). Again in the case of the mean field approach for a confined Bose gas, in [52] the aforementioned convergence problem is formulated and generalized within a McKean-Vlasov stochastic optimal control problem framework. All the cited results, both analytic and probabilistic, has a stationary character. Indeed, since the temperature is really very low, this first time-independent or ground state approach is physically justified. Nevertheless, recently there has been a growing interest in a time-dependent approach to BEC (see, e.g., [53], [54], [55], [56]). We recall that from the experimental point of view, the Bose-Einstein condensation is achieved by trapping a boson gas through a strong magnetic field and cooling it to temperatures close to absolute zero, and it is observed that in the condensate all the gas particles are described by the same wave function for a particle. One talks about time-dependent Bose-Einstein condensation when, once the condensation is obtained, the magnetic field cited above is instantly removed and the evolution over time of the condensate is observed. The goal of the present research project is twofold and involves both HCM at Bonn University and University of Pavia. The first research line is to generalize, in collaboration with Prof. Dr. Sergio Albeverio in Bonn and Dott. Francesco C. De Vecchi in Pavia, to the time-dependent case the results obtained in the stationary setting. A good starting point is to face the convergence in relative entropy of the involved probability laws. Unlike the stationary case, in the time-dependent case it is no longer possible to consider the ground state energies and to study the convergence of quantum energies for the N-body system to the limit energy associated with the Gross-Pitaevskii nonlinear functional. Indeed, the mathematical physical approach to time-dependent problems, due to Bardos et al. ([53], [54]), is no more based on quantum energies but on finite Schroedinger hierarchies and their convergence properties. Here the main mathematical objects are the density operator and its kernel, called the density matrix. Since the N-particle Schroedinger equation is linear, one can consider only pure states without loosing generality. The time evolution of the density operator is the Von Neumann equation. Density operators are trace class operators and one can introduce the marginal density operator as well as the marginal density matrices. It is notable that by assuming at the initial time a suitable symmetry property for the density matrix, due to indistinguishability of Boson particles, the Von Neumann equation preserves this symmetry property for successive times. The equation solved by the n-marginal density matrix is called the finite N-particle Schroedinger hierarchy. Sending N to infinity one can formally obtain an infinite Schroedinger hierarchy. By using an estimate on the kinetic energy of the N-particle system, established for the first time in [54] an important rigorous convergence result for the finite N-particle Schroedinger hierarchy was proved. More precisely, for every fixed time, its solution converges in weak* topology to a solution of the infinite Schroedinger hierarchy. A second relevant result is the stability of the infinite Schroedinger hierarchy and of the factorization of the marginal density matrix. The main point of the latter problem is to prove that if the limit of the N-particle distribution function is factorized at the initial time then it remains factorized also at all subsequent times. An equivalent and elegant formulation of Nelson's stochastic mechanics through a stochastic variational principle was provided by Guerra and Morato ([57]). They obtained a stochastic Hamilton-Jacobi equation, completely equivalent to the Schroedinger equation, given by two non-linear PDEs for two variables R and S, related to the real and complex part of the Schroedinger wave function, respectively. Furthermore, Carlen's theorem ([58]) to any solution to the Schroedinger equation rigorously associates a Nelson diffusion process, where the drift is expressed in terms of the two variables R and S. The equation for R, for example, is the well-known continuity equation. According to our point of view, the result of convergence of density matrices, in particular expressed in terms of integral kernels, will be fundamental in the proof of the convergence of relative entropy in the time-dependent case. A second research line of the present project concerns the identification of an effective and useful stochastic scheme associated to the Gross-Pitaevskii model of a BE condensate. In collaboration with Sergio Albeverio ([47]), by performing a Doob ground state transformation of the N-body Hamiltonian, the infinitesimal generator of the process corresponding to a Nelson diffusion with a density-dependent killing rate was derived. Our conjecture for this future work is that the process describing the motion of the single particle in the BE condensate could also be described by a Cox process with intensity function given by the square of the wave function of the condensate and driven by the given Nelson diffusion. Cox processes are very useful generalizations of Poisson point processes in which the intensity is allowed to be random, but depending continuously on time. This important class of point processes is well-studied for example in financial applications and the members of this class have the advantage that they can be easily numerically simulated. A stochastic description of BEC by means of a Cox process for a non-interacting gas was provided in [59] (see other references inside). We aim at generalizing to the interacting case this stochastic scheme in the GP scaling limit due to the transition to chaos property. The picture of the above stochastic description could be the following. In previous papers we proved that the motion of the single particle in a BE condensate can be useful described by a Nelson diffusion X (usually in three dimensions). Then by considering the random intensity as given by the image of X through the fixed time probability density of the Nelson process itself, when we condition on a particular realization of the cited random intensity, the jump process becomes a non-homogenous Poisson process with the given intensity. We first plan to study all the properties and consequences of this stochastic description and their consistency with our previous stochastic structure, for example, by exploiting the Feynman-Kac representation and the convergence properties of relative entropy. Finally, using probabilistic techniques on weakly interacting processes, we plan the study of the properties of large deviations and fluctuations of the one particle process in the GP scaling limit around the nonlinear limit process. For the corresponding results in the mathematical physics setting, based on operator theory, see, e.g, [60] and [61].

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